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LETTER TO THE EDITOR

# Application of a reciprocal transformation to a two-phase Stefan problem

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**Abstract.** A reciprocal transformation is employed to reduce a two-phase Stefan problem in nonlinear heat conduction to a form which admits a class of exact solutions analogous to the classical Neumann solution.

Storm (1951), in an investigation of heat transport in simple metals showed that for an important class of such materials a Bäcklund transformation may be introduced which reduces the governing nonlinear heat conduction equation to the classical $1+1$ heat equation. The method was used to solve a fixed boundary value problem involving a half-space with an insulated boundary. It has been shown recently that the Storm transformation may be set in the context of a class of reciprocal Bäcklund transformations which allow the reduction of a wide variety of nonlinear boundary value problems to linear canonical form (Rogers 1983). Here, a two-phase Stefan problem is considered for materials of Storm-type. Such moving boundary problems arise naturally in the analysis of melting and solidification processes (Rubinstein 1971). Their complexity resides in the fact that the heat balance condition at the moving interface separating the phases produces a nonlinear boundary condition. In the present problem there is the additional complication that the heat conduction equations that prevail on either side of the moving boundary are themselves nonlinear. It is shown that introduction of a reciprocal transformation allows the construction of a class of exact solutions analogous to the classical Neumann solution of linear heat conduction.

The two-phase Stefan problem considered is for a semi-infinite region $x>0$ with phase change temperature $T_f$. It is required to determine the evolution of the moving phase separation boundary $x=X(t)$ and temperature distribution

$$T(x,t)=\begin{cases} T_2(x,t)>T_f & 0<x<X(t) \\ T_f & x=X(t),\ t>0 \\ T_1(x,t)<T_f & X(t)<x<\infty \end{cases} \tag{1}$$

where

$$\rho c_{p1}(T_1)\,\partial T_1/\partial t=(\partial/\partial x)[\kappa_1(T_1)\,\partial T_1/\partial x], \qquad X(t)<x<\infty \tag{2}$$

$$\left.\begin{array}{l}\kappa_1(T_1)\,\partial T_1/\partial x-\kappa_2(T_2)\,\partial T_2/\partial x=L\rho\dot{X}\\ T_1=T_2=T_f\end{array}\right\}\text{on } x=X(t) \tag{3}$$

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$$\rho c_{p2}(T_2)\,\partial T_2/\partial t=(\partial/\partial x)[\kappa_2(T_2)\,\partial T_2/\partial x]m \qquad 0<x<X(t) \tag{4}$$

$$\kappa_2(T_2)\,\partial T_2/\partial x=U(t) \qquad \text{on } x=0,\ t>0 \tag{6}$$

together with the initial conditions

$$X(0)=0 \tag{6}$$

$$T_1(x,0)=V_0<T_f, \qquad x>0. \tag{7}$$

In the above, the $T_i(x, t)$, $c_{pi}(T_i)$, $\kappa_i(T_1)$, $i=1,2$ represent in turn the temperature distribution, specific heat and thermal conductivity in the two phases. The density $\rho$ of the medium is assumed to be constant. $L$ denotes the latent heat of fusion of the medium. In this problem a melting process is envisaged in which phase 1 is solid and phase 2 is liquid. Here $U(t)$ denotes the prescribed flux on the boundary $x=0$ while $V_0$ represents the initial temperature of the medium. It is noted that the analogous two-phase problem in linear heat conduction has been recently investigated by Tarzia (1982). As in that work attention is restricted to the class of moving boundary problems with

$$U(t)=U_0t^{1/2}, \qquad X(t)=(2\gamma t)^{1/2}. \tag{8,9}$$

If we now set

$$\bar{T}_i=\Phi_i(T_i)=\int_{T_{0i}}^{T_i} S_i(\sigma)\,\mathrm{d}\sigma, \qquad S_i=\rho c_{pi}(T_i) \qquad i=1,2 \tag{10}$$

then (2) and (4) become

$$\frac{\partial \bar{T}_i}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\kappa_i(T_i)}{\Phi_i'}\frac{\partial \bar{T}_i}{\partial x}\right)=0 \qquad i=1,2. \tag{11}$$

Our investigation is henceforth confined to materials for which

$$k_i\Phi_i'/\Phi_i^2=\kappa_i(T_i), \qquad i=1,2 \tag{12}$$

that is,

$$\frac{1}{(\kappa_iS_i)^{1/2}}\left[\ln\left(\frac{S_i}{\kappa_i}\right)^{1/2}\right]'=\frac{1}{k_i^{1/2}} \qquad i=1,2 \tag{13}$$

where the $k_i$, $i=1,2$ are positive constants. Conditions of the type (13) were originally obtained by Storm (1951) in an investigation of heat conduction in simple monatomic metals. There, the validity of the approximation was examined for aluminium, silver, sodium, cadium, zinc, copper and lead. It was shown that $\kappa S$ and $[\ln(S/\kappa)^{1/2}]'$ exhibit only small variation over wide temperature ranges within which, accordingly, the approximation (13) is justified. Similar conclusions were reached for iron and 80% carbon steel. A recent account of the experimental evidence on the variation of thermal conductivity and specific heat with temperature is given in Tslaf (1981).

Use of the conditions (12) in (11) reduces the heat conduction equations in the two phases to the form

$$\frac{\partial \bar{T}_i}{\partial t}-k_i\frac{\partial}{\partial x}\left(\frac{1}{\bar{T}_i^2}\frac{\partial \bar{T}_i}{\partial x}\right)=0 \qquad i=1,2. \tag{14}$$

The similarity variable

$$\xi=x(2\gamma t)^{-1/2} \tag{15}$$

is now introduced and solutions of (14) are sought of the type

$$\bar{T}_i = \phi_i x(2\gamma t)^{-1/2} \qquad i = 1, 2 \tag{16}$$

whence (14) yields

$$\gamma\xi\frac{\mathrm{d}\phi_i}{\mathrm{d}\xi} + k_i\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\phi_i^2}\frac{\mathrm{d}\phi_i}{\mathrm{d}\xi}\right) = 0 \qquad i = 1, 2. \tag{17}$$

Under the reciprocal transformation

$$\left.\begin{aligned} \mathrm{d}\xi &= \phi_i^* \,\mathrm{d}\xi_i^* \\ \phi_i^* &= \phi_i^{-1} \end{aligned}\right\} R \qquad (R^2 = I), \tag{18}$$

(17) becomes

$$-\gamma\xi\frac{\mathrm{d}}{\mathrm{d}\xi_i^*}\left(\frac{1}{\phi_i^*}\right) + k_i\frac{\mathrm{d}}{\mathrm{d}\xi_i^*}\left(\frac{1}{\phi_i^*}\frac{\mathrm{d}\phi_i^*}{\mathrm{d}\xi_i^*}\right) = 0, \tag{19}$$

whence, on integration

$$\frac{k_i}{\phi_i^*}\frac{\mathrm{d}\phi_i^*}{\mathrm{d}\xi_i^*} = \gamma\left(\frac{\xi}{\phi_i^*} - \xi_i^*\right) + C_i.$$

We set $C_i = 0$ without subsequent loss of generality to obtain

$$k_i \,\mathrm{d}\phi_i^*/\mathrm{d}\xi_i^* = \gamma(\xi - \phi_i^*\xi_i^*) \tag{20}$$

whence, on use of $R$ we obtain the *linear* canonical form

$$k_i \,\mathrm{d}^2\phi_i^*/\mathrm{d}\xi_i^{*2} + \gamma\xi_i^* \,\mathrm{d}\phi_i^*/\mathrm{d}\xi_i^* = 0 \qquad i = 1, 2 \tag{21}$$

with solution

$$\phi_i^* = A_i \,\mathrm{erf}\left[\left(\frac{\gamma}{2k_i}\right)^{1/2}\xi_i^*\right] + B_i \qquad i = 1, 2. \tag{22}$$

The four conditions

$$T_1 = T_2 = T_f \qquad \text{on } x = X(t)$$

$$\kappa_2(T_2)\,\partial T_2/\partial x = U_0 t^{-1/2} \qquad \text{on } x = 0,\ t > 0$$

$$T_1(x, 0) = V_0$$

produce, in turn, four equations which determine the $A_i$, $B_i$ $i = 1, 2$ namely

$$A_1 \,\mathrm{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2}\xi_1^*|_{\xi=1}\right] + B_1 = \frac{1}{\Phi_1(T_f)} \tag{23}$$

$$A_2 \,\mathrm{erf}\left[\left(\frac{\gamma}{2k_2}\right)^{1/2}\xi_2^*|_{\xi=1}\right] + B_2 = \frac{1}{\Phi_2(T_f)} \tag{24}$$

$$k_2 \,\mathrm{d}\phi_2^*/\mathrm{d}\xi_2^* = -U_0(2\gamma)^{1/2}\phi_2^* \qquad \text{on } \xi_2^* = \xi_2^*|_{\xi=0} \tag{25}$$

$$A_1 \,\mathrm{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2}\xi_1^*|_{\xi\to\infty}\right] + B_1 = \frac{1}{\Phi_1(V_0)}. \tag{26}$$

Now, on use of (20) together with (25) it is seen that

$$\xi_2^*|_{\xi=0} = U_0(2/\gamma)^{1/2} \tag{27}$$

whence, in general, (18) shows that

$$\xi_2^* = \int_0^{\xi} \phi_2(\sigma)\,\mathrm{d}\sigma + U_0(2/\gamma)^{1/2}. \tag{28}$$

Moreover, the interface condition

$$\kappa_1(T_1)\,\partial T_1/\partial x - \kappa_2(T_2)\,\partial T_2/\partial x = L\rho\dot{X} \qquad \text{on } x = X(t)$$

yields

$$-\frac{k_1}{\phi_1^*}\frac{\mathrm{d}\phi_1^*}{\mathrm{d}\xi_1^*} + \frac{k_2}{\phi_2^*}\frac{\mathrm{d}\phi_2^*}{\mathrm{d}\xi_2^*} = L\rho\gamma \qquad \text{on } \xi = 1$$

so that, from (20)

$$\xi/\phi_2^* - \xi_2^* - (\xi/\phi_1^* - \xi_1^*) = L\rho \qquad \text{on } \xi = 1. \tag{29}$$

Accordingly,

$$\Phi_2(T_f) - \Phi_1(T_f) + \xi_1^*|_{\xi=1} - \xi_2^*|_{\xi=1} = L\rho. \tag{30}$$

Now, $\phi_2$ is given parametrically in terms of $\xi$ via

$$\left.\begin{aligned} \phi_2 &= \left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2k_2}\right)^{1/2} \xi_2^*\right] + B_2\right\}^{-1} \\ \xi &= \int_{U(2/\gamma)^{1/2}}^{\xi_2^*} \left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2k_2}\right)^{1/2} \sigma\right] + B_2\right\} \mathrm{d}\sigma \end{aligned}\right\}. \tag{31}$$

On the other hand, $\phi_1$ is given parametrically in terms of $\xi$ via

$$\left.\begin{aligned} \phi_1 &= \left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2} \xi_1^*\right] + B_1\right\}^{-1} \\ \xi &= \int_{\xi^*|_{\xi=1}}^{\xi_1^*} \left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2} \sigma\right] + B_1\right\} \mathrm{d}\sigma + 1 \end{aligned}\right\}. \tag{32}$$

The boundary condition (25) and the initial condition (26) yield, in turn,

$$A_2(k_2/\pi)^{1/2} \exp(-U_0^2/k_2) = -U_0[A_2 \operatorname{erf}(U_0 k_2^{-1/2}) + B_2] \tag{33}$$

and

$$A_1 + B_1 = 1/\Phi_1(V_0). \tag{34}$$

To summarise the required temperature distributions $T_1$ and $T_2$ are given parametrically by

$$\left.\begin{aligned} T_1 &= \Phi_1^{-1}\left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2} \xi_1^*\right] + B_1\right\}^{-1}, \\ \xi &= \int_{\lambda_1}^{\xi_1^*} \left\{A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2} \sigma\right] + B_1\right\} \mathrm{d}\sigma + 1 \end{aligned}\right\} \tag{35}$$

and

$$\left.\begin{aligned} T_2 &= \Phi_2^{-1}\left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2} \xi_2^*\right] + B_2\right\}^{-1}, \\ \xi &= \int_{U_0(2/\gamma)^{1/2}}^{\xi_2^*} \left\{A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2k_2}\right)^{1/2} \sigma\right] + B_2\right\} \mathrm{d}\sigma \end{aligned}\right\}. \tag{36}$$

In the above, the $A_i$, $B_i$ $i=1,2$ are given in terms of $\lambda_1 = \xi_1^*|_{\xi=1}$ and $\lambda_2 = \xi_2^*|_{\xi=1}$ by the relations

$$\begin{aligned} &A_1 \operatorname{erf}\left[\left(\frac{\gamma}{2k_1}\right)^{1/2}\lambda_1\right] + B_1 = \frac{1}{\Phi_1(T_f)}, \\ &A_1 + B_1 = \frac{1}{\Phi_1(V_0)}, \\ &A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2k_2}\right)^{1/2}\lambda_2\right] + B_2 = \frac{1}{\Phi_2(T_f)} \\ &A_2(k_2/\pi)^{1/2}\exp(-U_0^2/k_2) = -U_0[A_2 \operatorname{erf}(U_0 k_2^{-1/2}) + B_2]. \end{aligned} \tag{37}$$

The quantities $\lambda_1$ and $\lambda_2$ are then given by (30), that is

$$\lambda_1 = L\rho + \Phi_1(T_f) - \Phi_2(T_f) + \lambda_2 \tag{38}$$

together with the relation

$$1 = \int_{U_0(2/\gamma)^{1/2}}^{\lambda_2} A_2 \operatorname{erf}\left[\left(\frac{\gamma}{2k_2}\right)^{1/2}\sigma\right] d\sigma + B_2[\lambda_2 - U_0(2/\gamma)^{1/2}]. \tag{39}$$

The constant $\gamma$ giving the speed of the moving boundary $x = X(t) = (2\gamma t)^{1/2}$ is determined by the nonlinear moving boundary condition which provides the transcendental equation

$$\begin{aligned} &-A_1\Phi_1(T_f)(2k_1/\pi)^{1/2}\exp(-\gamma\lambda_1^2/2k_1) \\ &\qquad + A_2\Phi_2(T_f)(2k_2/\pi)^{1/2}\exp(-\gamma\lambda_2^2/2k_2) = L\rho\gamma^{1/2}. \end{aligned} \tag{40}$$

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